Int. J. Heat Mass Transfer. Vol. 3, pp. 94–107. Pergamon Press 1961, Printed in Great Britain

TWO-DIMENSIONAL UNSTEADY INCOMPRESSIBLE LAMINAR DUCT FLOW WITH A STEP CHANGE IN WALL TEMPERATURE

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(*Received* 30 *September* 1960, *and in revised form* 6 *February* 1961)

Abstract—Analytical solutions are obtained of transient heat transfer for unsteady incompressible laminar flow between parallel plates. The transient is caused by simultaneously changing with time the driving pressure of the fluid and the wall temperature. The solution is first obtained for the case where the inside surfaces of the channel walls undergo a specified step in temperature, that is, the heat-transfer resistance of the wall is neglected. Then some results are given where the temperature is specified at the outside surfaces of the walls and the transient heat conduction through the walls is taken into account. A few numerical examples are carried out to illustrate the method.

Résumé—Des solutions analytiques sont obtenues pour la transmission de chaleur en régime transitoire, dans le cas d'un écoulement laminaire incompressible non permanent entre des plaques parallèles. Le régime transitoire est provoqué par des variations simultanées dans le temps de la pression motrice du fluide et de la température de paroi. La solution est tout d'abord obtenue dans le cas où les faces internes des parois du conduit subissent un saut donné de température, c'est-à-dire lorsque la résistance thermique de la paroi est négligée. Puis quelques résultats sont donnés dans le cas où la température des faces extérieures de la paroi est connue et où l'on tient compte de la conduction de chaleur transitoire à travers les parois. Quelques exemples numériques sont effectués pour illustrer la méthode.

Zusammenfassung—Für den veränderlichen Wärmeübergang einer nichtstationären inkompressiblen Laminarströmung zwischen parallelen Platten werden analytische Lösungen angegeben. Die Änderung wird dadurch hervorgerufen, dass man abhängig von der Zeit zugleich den Flüssigkeitsdruck und die Wandtemperatur variiert. Die Lösung ist zuerst für den Fall angegeben, dass die Innenfläche der Kanalwand eine bestimmte sprunghafte Temperaturänderung erleidet, der Wärmeleitwiderstand der Wand also vernachlässigt wird. Für einige weitere Ergebnisse ist die Temperatur auf die äussere Wandfläche bezogen und die veränderliche Wärmeleitung durch die Wand berücksichtigt. Einige numerische Beispiele veranschaulichen die Methode.

Аннотация—В статье дается аналитическое решение задачи неустановившегося теплопереноса для нестационарного ламинарного потока несжимаемой жидкости между параллельными пластинами. Неустановившееся состояние жидкости вызывается изменением давления жидкости и температуры стенки во времени. Задача решена для случаев, когда термическим сопротивлением стенки можно пренебречь, а также, когда его нужно учитывать. Выведенные расчётные формулы иллюстрируются численными примерами.

NOMENCLATURE

a, half-width of spacing between parallel plates;
b_{nm}, constants in series expansions of steady-state eigenfunctions, equation (6);
c_p, specific heat of fluid at constant pressure;
c_w, specific heat of channel walls;
d, thickness of channel walls;
E_i, eigenvalue, $(i + \frac{1}{2})\pi$; $E_m = (m + \frac{1}{2})\pi$;
F_n, function in problem where both pressure gradient and wall temperature change, equation (7);
f_n, constant defined in equation (15a);
G_n, function in problem where pressure gradient changes and wall temperature does not change, equation (18);
k, thermal conductivity of fluid;

k_w, thermal conductivity of wall;
Pr, Prandtl number of fluid, $c_p\mu/k = \nu/\alpha$;
p, static pressure;
q, local heat transferred per unit area from channel walls to fluid;
Re, Reynolds number, $\bar{u}4a/\nu$;
S, friction coefficient defined by equation (3);
s_w, shear stress at wall;
T, dimensionless temperature, $(t - t_0)/(t_w - t_0)$;
t, temperature;
t_0, temperature of fluid entering channel (a constant);
t_w, temperature at inside surface of wall in contact with fluid;
$t_{w,1}$, specified wall temperature before transient;
$t_{w,2}$, specified wall temperature during transient;
t_w^*, temperature at outside surface of wall;
u, fluid velocity;
$\bar{u}_1$, mean velocity before transient;
$\bar{u}_2$, final mean velocity after transient;
X, dimensionless co-ordinate, $8x/3a\,RePr$;
X_0, value of X at $\Theta = 0$ at beginning of a characteristic line;
x, axial distance from start of heated section of channel;
Y, dimensionless co-ordinate, y/a;
y, transverse co-ordinate measured from centerline of channel.

Greek symbols
α, thermal diffusivity of fluid, $k/\rho c_p$;
α_w, thermal diffusivity of wall, $k_w/\rho_w c_w$;
β_n^2, constant defined in equation (16);
Θ, dimensionless time, $\tau\nu/a^2 Pr$;
Θ_0, value of Θ at $X = 0$ at beginning of a characteristic line;
θ, dimensionless time, $\tau\nu/a^2$;
λ_n^2, steady-state eigenvalue;
μ, absolute viscosity;
ν, kinematic viscosity;
ρ, fluid density;
ρ_w, density of wall;
σ_n, constant defined in equation (11a);
τ, time;
ψ_n, expansion for steady-state eigenfunction, equation (7a); average values,

$$\bar{\psi}_n = \int_0^1 \psi_n \, dY, \quad \overline{\frac{u}{\bar{u}_2}\psi_n} = \int_0^1 \frac{u}{\bar{u}_2}\psi_n \, dY.$$

INTRODUCTION

Unsteady internal flows with unsteady heat transfer are encountered in a wide variety of heat transfer devices. Some examples are the starting of a rocket engine, shutdown of a nuclear reactor, or during changes in propulsive power of a vehicle powerplant. There have been a number of papers dealing with unsteady heat transfer to flows in tubes and ducts, and [1] and [2] provide several references on this subject. These papers have been restricted to situations where the fluid velocity does not vary with time. For unsteady velocities, some information for flow in a circular tube with a constant wall temperature is given in [3].

In a previous paper [4] the authors considered heat transfer in the thermal entrance and fully developed regions for unsteady flow between parallel plates where the walls had either a constant temperature or had a uniform heat flux transferred from them. The transients were initiated by simultaneously changing with time the fluid pumping pressure and either the wall temperature or the wall heat flux. In [4] the problem was simplified by using a one-dimensional (slug flow) approximation for the velocity distribution in the channel. Within the limitation of this assumption, exact solutions for the fluid temperature distributions were obtained. One objective in the present work is to try to account for the variation in velocity across the channel cross section. A second objective is concerned with the heat conduction through the channel walls. In [4] the heat transfer resistance of the channel walls was neglected so that the thermal boundary conditions were assigned at the inside surfaces of the walls where they are in contact with the fluid. This restriction will be discussed in a later section of the paper, and some results for a finite wall resistance will be given.

The configuration selected for analysis is shown in Fig. 1. It consists of two parallel plates of thickness d with incompressible laminar flow between them. An unheated hydrodynamic entrance region is provided between the entrance of the channel and the start of the heated section, so that within the heated section the velocity is no longer a function of axial position along the channel length. The first problem that is considered is the transient resulting from a step change in time of both the

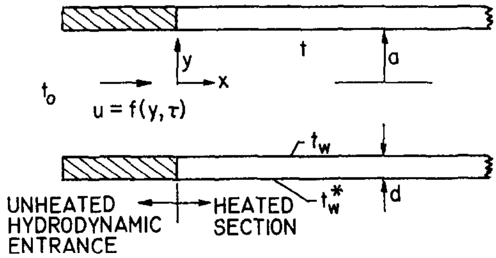

FIG. 1. Parallel-plate channel.

fluid pumping pressure and the temperature at the inside surface of the wall, t_w. The method used does not provide an exact solution of the governing partial differential equation, but involves an integral approximation at one step in the analysis. The validity of this approximation is tested by comparisons with exact results that are available for part of the solution. Then the results are extended to consider a change in temperature at the outside surface of the wall, t_w^*, and illustrative examples are given.

TRANSIENT VELOCITY DISTRIBUTION

Since the convective term in the energy equation contains the fluid velocity, the transient velocity must first be determined. The equation of motion for fully developed incompressible laminar flow between parallel plates is given by

$$\frac{\partial u}{\partial \tau} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

where $\partial p/\partial x$ is a function of time. In the transient considered here, the fluid is moving initially with a steady mean velocity $\bar{u}_1$. Then the driving pressure difference is suddenly changed so that the velocity adjusts to a new mean value $\bar{u}_2$. The solution for the transient has been given in [4] as

$$\frac{u}{\bar{u}_2} = \frac{3}{2}(1 - Y^2) - 6\left(1 - \frac{\bar{u}_1}{\bar{u}_2}\right)\sum_{i=0}^{\infty}\frac{(-1)^i}{E_i^3}\exp(-E_i^2\theta)\cos E_i Y \tag{2}$$

where the dimensionless variables are defined in the nomenclature. In this solution the time required for the step pressure change to be transmitted throughout the fluid is neglected compared with the duration of velocity adjustment. The final pressure gradient imposed in the channel does not appear explicitly because it has been taken into account by the steady-state relation,

$$\bar{u}_2 = -\frac{a^2}{3\mu}\left(\frac{dp}{dx}\right)_2.$$

Equation (2) will apply for all transients except those where the final velocity $\bar{u}_2$ is zero, which occurs when the pressure gradient is dropped to zero. For this case we multiply equation (2) by $\bar{u}_2/\bar{u}_1$ and then let $\bar{u}_2 = 0$ to obtain the result,

$$\frac{u}{\bar{u}_1} = 6\sum_{i=0}^{\infty}\frac{(-1)^i}{E_i^3}\exp(-E_i^2\theta)\cos E_i Y. \tag{2a}$$

Equation (2a) has been plotted as a function of Y in Fig. 2 for various θ values. From these results all other transient velocity distributions can be found by using equation (2).

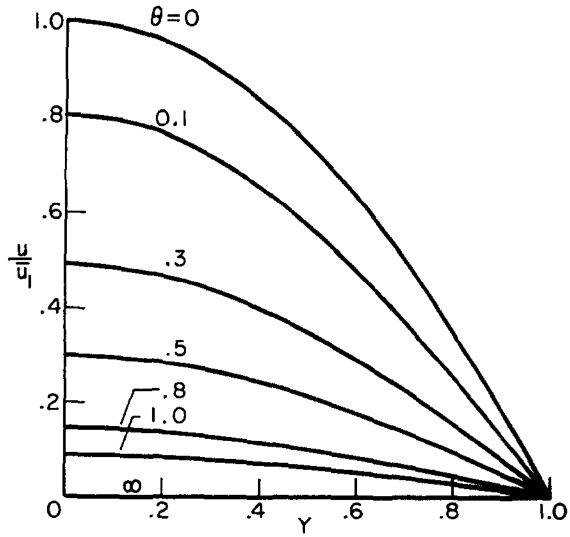

FIG. 2. Transient velocity profiles for case where pressure gradient is dropped to zero. These can be used to find all other profiles by using equation (2).

Wall friction

During the velocity transient, part of the pressure drop is used to change the momentum of the fluid and the remainder is used to overcome the fluid friction. As a matter of general interest we shall look at how the wall friction deviates from the steady value during the transient. A wall friction coefficient for laminar flow is defined as

$$S = \frac{s_w a}{\mu \bar{u}} \tag{3}$$

where s_w is the shear stress at the wall;

$$s_w = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=a},$$

and $\bar{u}$ is the instantaneous mean velocity. By using the velocity as given in equation (2) we obtain the expression

$$S = \frac{1 - 2\left(1 - \dfrac{\bar{u}_1}{\bar{u}_2}\right)\displaystyle\sum_{i=0}^{\infty} \frac{1}{E_i^2} \exp(-E_i^2\theta)}{\dfrac{1}{3} - 2\left(1 - \dfrac{\bar{u}_1}{\bar{u}_2}\right)\displaystyle\sum_{i=0}^{\infty} \frac{1}{E_i^4} \exp(-E_i^2\theta)}. \tag{3a}$$

This is plotted as a function of θ in Fig. 3 for various pressure gradient ratios. For steady flow the value of S is 3. For a flow that is being accelerated $(dp/dx)_2 > (dp/dx)_1$, the friction coefficient rises above that for fully developed flow and then decreases to the steady value. This is caused by the fact that the fluid near the wall has a small momentum because of its low velocity and hence responds more quickly when the pressure force is changed. As a result, during the early part of the transient, the wall shear stress approaches the final value more rapidly than the mean fluid velocity, resulting in higher instantaneous values of the ratio S. During a deceleration the derivative at the wall decreases more rapidly than the mean velocity, and hence the friction coefficient goes through a minimum.

TRANSIENT HEAT TRANSFER FOR ZERO WALL RESISTANCE

The transient heat transfer results given in this paper are caused by a step change with time of the wall temperature in the heated section of the channel. In this section we consider a wall that is either very thin or has a very high thermal diffusivity so that the inside surface of the wall in contact with the fluid instantaneously reaches the value of the imposed wall temperature. The most general solution for this case is formed by superposing two more elementary results. These will be given first, and then the superposition will be described.

Step change in time of both pressure gradient and wall temperature from an unheated initial condition

Before the transient begins, the fluid is moving in a steady fashion with a mean velocity $\bar{u}_1$ that can also be zero as a special case. The walls of the channel are unheated, and the whole system is at the entering fluid temperature t_0. Then the pressure gradient in the channel is abruptly changed so that the fluid velocity undergoes a transient to a new mean velocity $\bar{u}_2$. At the same instant that the pressure gradient is changed, the temperature at the inside surface of the wall in the heated section of the channel is changed to a new value $t_{w,2}$. It is desired to compute how the heat transfer to the fluid varies with time and location along the channel.

In the analysis it is assumed that the fluid has constant properties. Axial heat conduction and viscous dissipation are neglected compared with heat conduction in the direction across the channel. With these restrictions, the energy equation for forced convection in the channel can be written as

$$\frac{\partial t}{\partial \tau} + u\frac{\partial t}{\partial x} = \alpha \frac{\partial^2 t}{\partial y^2}. \tag{4}$$

The unsteady velocity distribution equation (2) is then inserted, and the resulting equation is placed in the following dimensionless form:

$$\frac{\partial T}{\partial \Theta} + \left[1 - Y^2 - 4\left(1 - \frac{\bar{u}_1}{\bar{u}_2}\right) \times \sum_{i=0}^{\infty} \frac{(-1)^i}{E_i^3} \exp(-E_i^2 Pr\Theta) \cos E_i Y\right] \frac{\partial T}{\partial X} = \frac{\partial^2 T}{\partial Y^2}. \tag{4a}$$

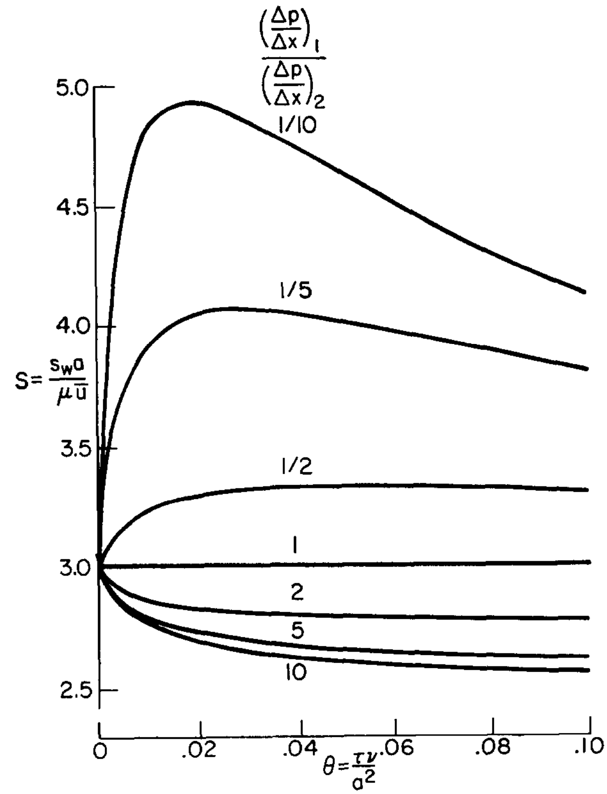

FIG. 3(a) Friction coefficients as a function of time for various pressure-gradient ratios.

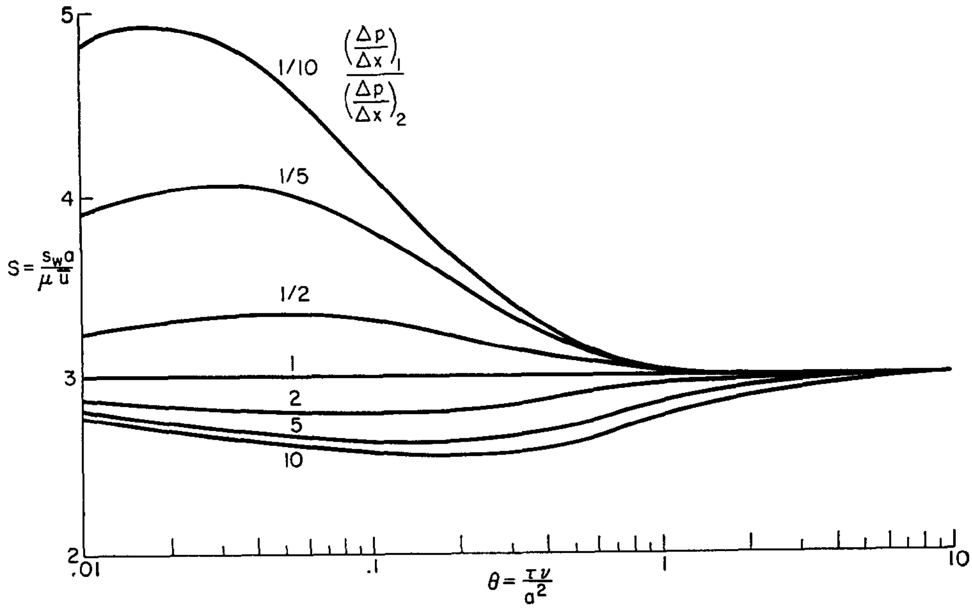

FIG. 3(b) Friction coefficients as a function of time for various pressure-gradient ratios.

This equation is to be solved subject to the boundary conditions:

$T = 0$ at $X = 0$ for all Y and Θ, entrance condition (5a)

$T = 0$ at $\Theta = 0$ for all Y and X, initial condition (5b)

$T = 1$ at $Y = 1$ for all X and for $\Theta > 0$, specified wall temperature (5c)

$\frac{\partial T}{\partial Y} = 0$ at $Y = 0$ for all X and Θ, symmetry (5d)

To obtain a solution to equation (4a), we first consider the steady-state solution in the thermal entrance region for flow between parallel plates with a constant surface temperature. One form of this solution has been given in [2] as

$$T_s = 1 - \sum_{n=0}^{4}\left(\sum_{m=0}^{4}\frac{b_{nm}}{b_{n0}}\cos E_m Y\right) b_{n0}\exp(-\lambda_n^2 X). \quad (6)$$

The summation in parenthesis is a series expansion for the eigenfunctions that arise from the product solution of the steady-state partial differential equation. The coefficients b_{nm}/b_{n0} and b_{n0}, and the eigenvalues λ_n^2 are given in Table 1 for a five-term expansion. This form of the steady solution is convenient for the present analysis as it enables the transient solutions to be carried out analytically rather than numerically.

Following the method given in [2], a transient solution is tried of the same general form as equation (6):

$$T = 1 - \sum_{n=0}^{4} b_{n0}\,\psi_n(Y)\,F_n(X,\Theta) \quad (7)$$

where for convenience we let

$$\psi_n(Y) = \sum_{m=0}^{4}\frac{b_{nm}}{b_{n0}}\cos E_m Y. \quad (7a)$$

Equation (7) already satisfies the boundary conditions (5c) and (5d). When Θ is very large, F_n should converge to the steady result, $\exp(-\lambda_n^2 X)$. To obtain the F_n, an approximation is made that the transient solution is only required to satisfy an integrated form of the energy equation. The validity of this approximation will be discussed a little later. The integrated form of equation (4a) is

$$\frac{\partial}{\partial\Theta}\int_0^1 T\,\mathrm{d}Y + \frac{2}{3}\frac{\partial}{\partial X}\int_0^1 \frac{u}{\bar{u}_2}T\,\mathrm{d}Y = \left.\frac{\partial T}{\partial Y}\right|_{Y=1}. \quad (8)$$

The trial solution equation (7) is substituted into equation (8), which yields a partial differential equation for F_n:

Table 1. Coefficients in five-term approximation for flow between parallel plates

n	b_{n0}	b_{n1}/b_{n0}	b_{n2}/b_{n0}	b_{n3}/b_{n0}	b_{n4}/b_{n0}
0	1·17776	0·0211834	−0·00113896	0·000207037	−0·0000586707
1	0·0579815	−5·37195	−0·838971	0·0230766	−0·00930795
2	0·0165696	−4·00649	9·45808	3·31526	0·101483
3	0·00706883	−3·59673	7·94700	−12·0113	−7·12402
4	0·0138607	−3·32432	6·61879	−11·0745	13·7624

n	λ_n^2	σ_n	β_n^2	f_n
0	2·82776	2·30858	2·82948	1·22564
1	32·1475	11·9441	32·3656	2·70975
2	93·4792	24·9156	95·2824	3·82421
3	187·388	37·4291	178·074	4·75764
4	414·761	92·5592	608·811	6·57754

$$\frac{\partial F_n}{\partial \Theta}\int_0^1 \psi_n \, \mathrm{d}Y + \frac{2}{3}\frac{\partial F_n}{\partial X}\int_0^1 \frac{u}{\bar{u}_2}\psi_n \, \mathrm{d}Y = F_n \frac{\mathrm{d}\psi_n}{\mathrm{d}Y}\bigg|_{Y=1}. \tag{9}$$

This type of equation can be treated by using the method of characteristics (Ref. 5, p. 371). According to this method, the following set of auxiliary ordinary differential equations is formed from the coefficients in equation (9):

$$\frac{\mathrm{d}\Theta}{\bar{\psi}_n} = \frac{\mathrm{d}X}{\overline{\frac{2}{3}\frac{u}{\bar{u}_2}\psi_n}} = \frac{\mathrm{d}F_n}{F_n \frac{\mathrm{d}\psi_n}{\mathrm{d}Y}\Big|_{Y=1}} \tag{10}$$

where a bar has been used to abbreviate the integral notation (e.g. $\bar{\psi}_n = \int_0^1 \psi_n \, \mathrm{d}Y$). If the first two terms of equation (10) are integrated, equations for characteristic curves on the X,Θ plane are obtained as shown schematically in Fig. 4. There is a different set of characteristic curves for each value of n. The characteristics beginning at the origin divide the plane into two regions: In region I the curves originate at the X axis, while in region II they originate from the Θ axis. By equating the last term in equation (10) with either of the other two terms, an equation for the F_n variation along the characteristic curves is obtained.

To determine F_n in region I, the first and last terms in equation (10) are equated and integrated:

$$\int_1^{F_n} \frac{\mathrm{d}F_n}{F_n} = -\,\sigma_n \int_0^{\Theta} \mathrm{d}\Theta \tag{11}$$

where

$$\sigma_n = -\frac{1}{\bar{\psi}_n}\frac{\mathrm{d}\psi_n}{\mathrm{d}Y}\bigg|_{Y=1} = \frac{\sum_{m=0}^{4} E_m(-1)^m \frac{b_{nm}}{b_{n0}}}{\sum_{m=0}^{4} \frac{(-1)^m}{E_m}\frac{b_{nm}}{b_{n0}}}. \tag{11a}$$

Numerical values of σ_n are given in Table 1. The condition that $F_n = 1$ at $\Theta = 0$ is imposed to fulfill the initial condition (5b) which is satisfied because $1 - \sum_{n=0}^{4} b_{n0}\psi_n(Y) = 0$ as given by equation (6) at $X = 0$. The indicated integration in equation (11) is carried out to yield F_n in region I:

$$F_n = \exp(-\sigma_n \Theta). \tag{12}$$

To determine F_n in region II, the first and last terms in equation (10) are equated and then integrated along a region II characteristic curve:

$$\int_1^{F_n} \frac{\mathrm{d}F_n}{F_n} = -\,\sigma_n \int_{\Theta_0}^{\Theta} \mathrm{d}\Theta. \tag{13}$$

The boundary condition $F_n = 1$ at $\Theta = \Theta_0$, where $\Theta = \Theta_0$ corresponds to $X = 0$ on the characteristic curve, fulfills the entrance condition (5a). Integrating yields

$$F_n = \exp[-\sigma_n(\Theta - \Theta_0)]. \tag{14}$$

The arbitrary starting value Θ_0 must now be eliminated, and this is accomplished by utilizing

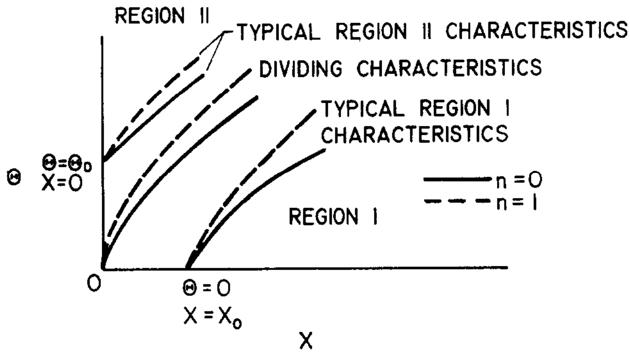

FIG. 4. Characteristic lines on X-Θ plane.

the equation of the characteristic curves originating at Θ_0. This is found by integrating the first two terms in equation (10):

$$\int_0^X \mathrm{d}X = \frac{2}{3}\int_{\Theta_0}^{\Theta} \frac{\frac{\overline{u}}{\bar{u}_2}\psi_n}{\bar{\psi}_n}\,\mathrm{d}\Theta. \tag{15}$$

After carrying out the integrations, this becomes

$$X = \frac{\Theta - \Theta_0}{f_n} + \frac{2\left(1 - \frac{\bar{u}_1}{\bar{u}_2}\right)}{Pr\,\bar{\psi}_n}\sum_{m=0}^{4}\frac{(-1)^m}{E_m^5}\frac{b_{nm}}{b_{n0}}$$
$$\times[\exp(-E_m^2 Pr\Theta) - \exp(-E_m^2 Pr\Theta_0)] \tag{15a}$$

where

$$f_n = \frac{\bar{\psi}_n}{\overline{(1 - Y^2)\psi_n}} = \frac{\sum_{m=0}^{4}\frac{(-1)^m}{E_m}\frac{b_{nm}}{b_{n0}}}{2\sum_{m=0}^{4}\frac{(-1)^m}{E_m^3}\frac{b_{nm}}{b_{n0}}}$$

and

$$\bar{\psi}_n = \sum_{m=0}^{4}\frac{(-1)^m}{E_m}\frac{b_{nm}}{b_{n0}}.$$

Numerical values of f_n are given in Table 1. Equation (14) is then solved for Θ_0, and this is substituted into equation (15a) to yield the following implicit expression for F_n as a function of X and Θ in region II:

$$X = -\frac{\log_e F_n}{\beta_n^2} + \frac{2\left(1 - \frac{\bar{u}_1}{\bar{u}_2}\right)}{Pr\,\bar{\psi}_n}\sum_{m=0}^{4}\frac{(-1)^m}{E_m^5}\frac{b_{nm}}{b_{n0}}$$
$$\times \exp(-E_m^2 Pr\Theta)\left[1 - \exp\left(-E_m^2 Pr\frac{\log_e F_n}{\sigma_n}\right)\right] \tag{16}$$

where

$$\beta_n^2 = f_n\sigma_n = -\frac{1}{\overline{(1 - Y^2)\psi_n}}\left.\frac{\mathrm{d}\psi_n}{\mathrm{d}Y}\right|_{Y=1}$$

The values of β_n^2 are given in Table 1. At steady state, since Θ is very large, the solution is found in region II, and equation (16) applies. When Θ is very large, the exponential term drops out and the equation reduces to

$$F_n = \exp(-\beta_n^2 X).$$

The steady-state solution equation (6) gave $F_n = \exp(-\lambda_n^2 X)$, and it is shown in Table 1 that the β_n^2 values are in good agreement with the λ_n^2 except for large n. If a larger number of terms were used for the series expansion in equation (6), then the agreement would improve, since the series approximation, ψ_n, for the eigenfunction would more closely approximate the exact function. As discussed in [2], if the exact eigenfunctions had been used, then β_n^2 would equal λ_n^2. Hence, within the limitation of the five-term approximation used here, the solution equation (16) approaches the exact solution at large time. The steady state results for the five-term approximation agree within a few per cent with the results in [6] and [7].

We can now summarize the solution. For each value of n, the X,Θ plane is divided into two regions by the characteristic curve passing through the origin. This dividing curve is given by equation (15a) with Θ_0 set equal to zero. For early times so that the solution is found in region I (Fig. 4) below the dividing characteristic, equation (12) is used for F_n. For later times so that the solution is found in region II above the dividing curve, equation (16) is used for F_n. For each F_n there is a different dividing characteristic curve. The F_n are then summed according to equation (7) to yield the transient temperature distribution.

The heat transferred from the wall to the fluid is obtained from Fourier's law:

$$q = k\left.\frac{\partial t}{\partial y}\right|_{y=a}.$$

By differentiating equation (7) and substituting into this relation, the heat flow becomes

$$\frac{qa}{k(t_{w,2} - t_0)} = -\sum_{n=0}^{4}\sum_{m=0}^{4} b_{n0}\frac{b_{nm}}{b_{n0}}E_m(-1)^m F_n. \tag{17}$$

To provide an illustrative example, this expression has been evaluated for a case when $\bar{u}_1 = 0$; that is, initially there is no flow and both the channel and fluid are isothermal at t_0. Then a pumping pressure is suddenly applied, and simultaneously the wall temperature in the heated section is stepped to $t_{w,2}$. The resultant transient wall heat transfer for a fluid with $Pr = 0{\cdot}7$ is shown in Fig. 5. At each axial location the heat transfer goes through a minimum and then rises to the steady-state value. This type of behavior has been previously demonstrated in [4], where a slug-flow approximation was used for the velocity distribution. As pointed out in [2], for early times the heat transfer results for the example treated here can be predicted from the transient heat-conduction equation. This is due to the fact that after the initiation of the transient, the heat transfer at a given location proceeds as if the tube were of infinite length until fluid that was outside the entrance of the heated section at the start of the transient reaches that location. For this early part of the transient, there is no variation in temperature in the axial direction, and the convection term drops out of the energy equation. Hence, if the solution were exact, the curve moving downward at the left side of Fig. 5 would be in agreement with the transient heat conduction into a solid slab of thickness $2a$, which is initially isothermal at temperature t_0 and then receives a step in the surface temperature. This conduction solution is shown dashed in the figure, and the approximate solution is in good agreement with it. Thus, the transient solution obtained here yields good results for small times and also for large times as discussed earlier.

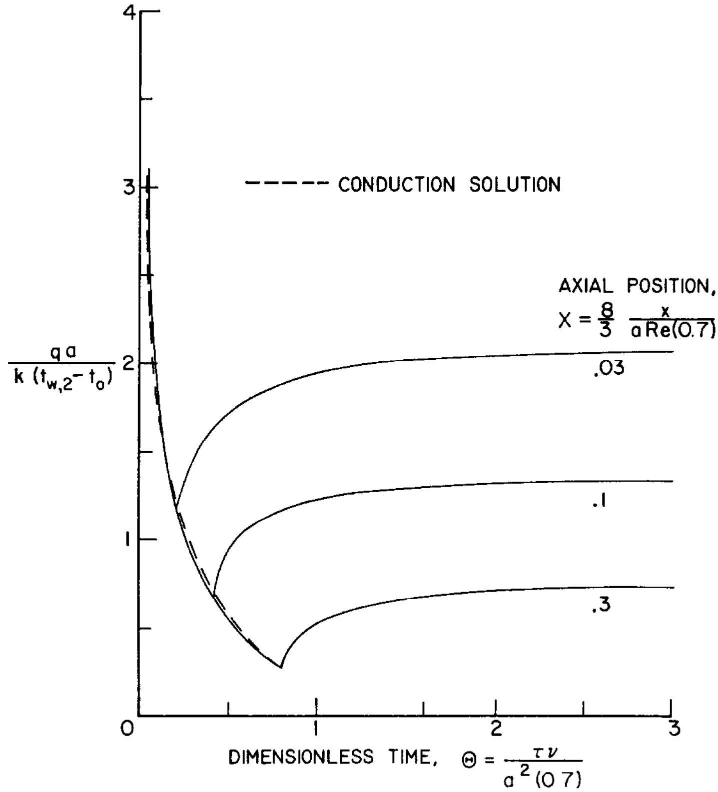

FIG. 5. Transient wall heat flux after a step change in pressure gradient and wall temperature. $Pr = 0{\cdot}7$; initially: $u = u_1 = 0$, $t_w = t_{w,1} = t_0$; finally: $u = u_2$, $t_w = t_{w,2}$.

Step change in pressure gradient with no change in wall temperature

Another type of transient will now be considered that can be superposed with the results in the previous section to solve a very general case. The channel walls in the heated section are initially at a constant temperature different from that in the isothermal hydrodynamic entrance region, and there is a steady-state heat transfer. Then the pumping pressure is suddenly changed so that the flow velocity undergoes a transient change to a new steady value. Throughout the process the wall temperature is maintained at its original value.

A solution is tried that has the same form as equation (7):

$$T = 1 - \sum_{n=0}^{4} b_{n0}\psi_n(Y)\, G_n(X,\Theta). \tag{18}$$

The boundary conditions to be satisfied are the same as equations (5), except for the initial condition (5b). Since at time zero there is a steady heat transfer taking place, we have from the steady solution equation (6) that

$$G_n = \exp\,[-\lambda_n^2(\bar{u}_2/\bar{u}_1)X] \text{ at } \Theta = 0, \quad \text{initial condition.} \tag{19}$$

The ratio $\bar{u}_2/\bar{u}_1$ appears because X has been non-dimensionalized on the basis of $\bar{u}_2$, while at $\Theta = 0$ the fluid mean velocity is $\bar{u}_1$. The trial solution equation (18) is substituted into the integrated energy equation (8), and the resulting partial differential equation yields the same auxiliary ordinary equations as in equation (10), with F_n being replaced by G_n. Since the only boundary condition that has been changed is the initial condition, the expressions for the characteristic passing through the origin and for G_n in region II are the same as those for the F_n in the previous section. Hence, we only have to be concerned with region I.

For a characteristic curve in region I, the first two terms of equation (10) are integrated starting from a point $\Theta = 0$, $X = X_0$. This gives

$$X - X_0 = \frac{\Theta}{f_n} + \frac{2\left(1 - \dfrac{\bar{u}_1}{\bar{u}_2}\right)}{Pr\,\bar{\psi}_n} \times \sum_{m=0}^{4} \frac{(-1)^m}{E_m^5}\frac{b_{nm}}{b_{n0}}\,[\exp\,(-E_m^2\,Pr\Theta) - 1]. \tag{20}$$

To obtain the variation in G_n along a characteristic in region I the first and third terms of equation (10) are integrated as follows:

$$\int^{G_n} \frac{\mathrm{d}G_n}{\exp\,[-\lambda_n^2(\bar{u}_2/\bar{u}_1)X_0]\,G_n} = -\sigma_n \int_0^{\Theta} \mathrm{d}\Theta. \tag{21}$$

This gives

$$G_n = \exp\,[-\sigma_n\Theta - \lambda_n^2(\bar{u}_2/\bar{u}_1)X_0]. \tag{21a}$$

This is solved for X_0, and the result is substituted into equation (20) to eliminate the arbitrary starting point X_0. The final expression for G_n in region I then becomes

$$\frac{1}{\lambda_n^2}\frac{\bar{u}_1}{\bar{u}_2}\log_e G_n = -X - \frac{\sigma_n}{\lambda_n^2}\frac{\bar{u}_1}{\bar{u}_2}\Theta + \frac{\Theta}{f_n} + \frac{2\left(1 - \dfrac{\bar{u}_1}{\bar{u}_2}\right)}{Pr\,\bar{\psi}_n}\sum_{m=0}^{4}\frac{(-1)^m}{E_m^5}\frac{b_{nm}}{b_{n0}} \times [\exp\,(-E_m^2 Pr\Theta) - 1]. \tag{22}$$

Now that the G_n are known, we can use equation (17) with G_n substituted for F_n to evaluate a numerical example. The example chosen is where the pressure gradient is given a step from an initially zero value. This means that, before the transient begins, there is no flow; and hence the heated section of the channel is filled with fluid at the wall temperature, which is different from the outside fluid temperature, and there is no heat transfer taking place. When the flow begins there is a transient period during which the fluid in the heated section of the channel is being swept out by fluid entering at t_0. During this early transient period no heat is being transferred. After this period, which occupies region I, heat flow begins, and the heat transfer rises toward the steady value. The results for a few axial locations and for $Pr = 0{\cdot}7$ are shown in Fig. 6. The slight discontinuities in derivative on the curve for $X = 0{\cdot}03$ occur where successive terms of the series in equation (17) are added together.

Step change in both pressure gradient and wall temperature with initial steady heating

The results given in the two previous sections can now be superposed to solve a more general situation. In this instance, there is initially a

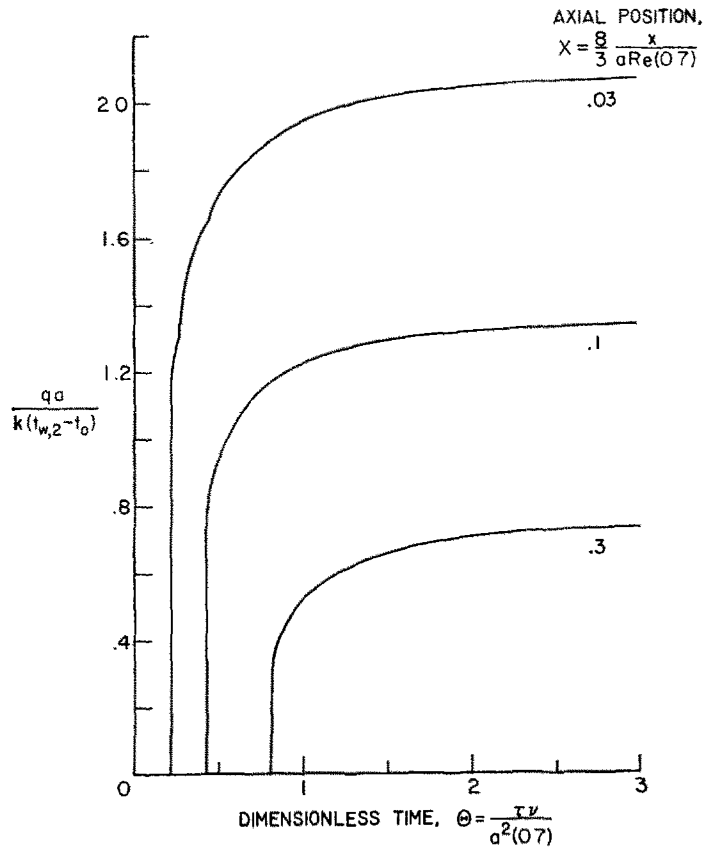

FIG. 6. Wall heat transfer after a step change in pressure gradient with the wall temperature kept constant ($t_{w,2} = t_{w,1}$). $Pr = 0{\cdot}7$; initially: $u = u_1 = 0$; finally: $u = u_2$.

steady heat transfer taking place. Then both the wall temperature in the heated section and the pressure gradient are suddenly changed to new values. The superposition used for this solution is illustrated in Fig. 7. The solution for the first part of the figure is given by the results of the first of the two preceding sections:

$$t = (t_{w,2} - t_{w,1})\,[1 - \sum_{n=0}^{4} b_{n0}\psi_n(Y)\,F_n(X,\Theta)].$$

The solution for the second part of the figure is given by the results in the preceding section:

$$t = (t_{w,1} - t_0)\,[1 - \sum_{n=0}^{4} b_{n0}\psi_n(Y)\,G_n(X,\Theta)] + t_0.$$

These results are then added to yield the general solution:

$$\frac{t - t_0}{t_{w,2} - t_0} = \left(\frac{t_{w,2} - t_{w,1}}{t_{w,2} - t_0}\right) \times [1 - \sum_{n=0}^{4} b_{n0}\psi_n(Y)\,F_n(X,\Theta)] + \left(\frac{t_{w,1} - t_0}{t_{u,2} - t_0}\right)[1 - \sum_{n=0}^{4} b_{n0}\psi_n(Y)\,G_n(X,\Theta)]. \quad (23)$$

TRANSIENT HEAT TRANSFER FOR FINITE WALL RESISTANCE

For the solutions in the previous section, the temperature t_w at the inside surface of the wall

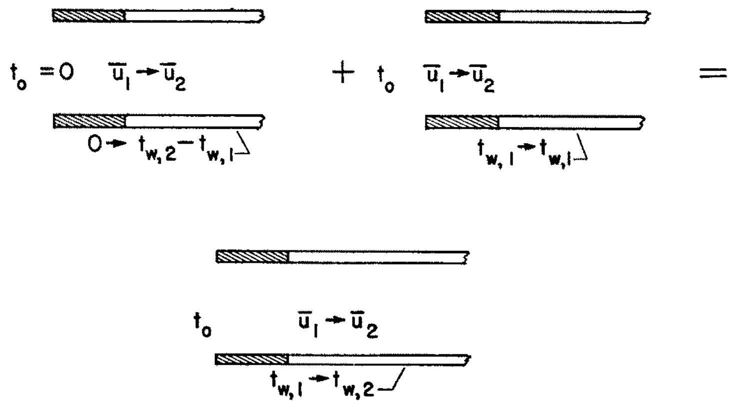

FIG. 7. Superposition of solutions for changing both pressure gradient and wall temperature.

was specified in the boundary conditions. We would now like to investigate the influence of a channel wall of finite thickness and finite thermal diffusivity where the temperature t_w^* at the outside surface is specified. In what follows we shall obtain part of the transient solution, and this will give an indication of how the additional factor of heat transport through the wall influences the transient response in the fluid.

The situation considered is as follows. The channel walls and fluid are initially isothermal at t_0 and there is no flow. Then both the outside surface temperature of the wall t_w^* in the heated section and the pressure gradient are simultaneously stepped to new values. As discussed previously, there follows an initial transient period of pure conduction in the fluid, and this continues at a given location until fluid that was originally outside of the heated section when the transient began reaches that location. Hence, for this initial period, if free convection is neglected the problem can be treated by considering the transient conduction through a two-layer slab. The first layer is the channel wall, and the second layer is the fluid contained between the wall and the center of the channel cross section. The boundary conditions are that the outside surface of the composite slab is suddenly given a step in temperature while the inside surface is kept perfectly insulated, which corresponds to the zero derivative in the fluid temperature distribution at the channel centerline. An analytical solution for this transient-conduction solution has been given in [8]. The heat transferred from the inside surface of the wall to the fluid is given by the relation

$$\frac{qa}{k(t_w^* - t_0)} = \frac{k_w a}{kd} \sum_{n=1}^{\infty} \frac{\rho_w c_w N_{1n} + \rho c_p N_{2n}}{\rho_w c_w D_{1n} + \rho c_p D_{2n}} \times (\Lambda_{1n} \cot \Lambda_{1n}) \exp(-\delta_n^2 \tau) \quad (24)$$

where the coefficients are found from

$$N_{1n} = \frac{d}{\Lambda_{1n}} [\sin \Lambda_{1n} + \cot \Lambda_{1n}(\cos \Lambda_{1n} - 1)]$$

$$N_{2n} = \frac{a}{\Lambda_{2n}} [\sin \Lambda_{2n} - \tan \Lambda_{2n}(\cos \Lambda_{2n} - 1)]$$

$$D_{1n} = \frac{d}{2\Lambda_{1n}} [(1 + \cot^2\Lambda_{1n}) \Lambda_{1n} + (1 - \cot^2\Lambda_{1n}) \sin \Lambda_{1n} \cos \Lambda_{1n} - 2 \cot \Lambda_{1n} \sin^2\Lambda_{1n}]$$

$$D_{2n} = \frac{a}{2\Lambda_{2n}} [(1 + \tan^2\Lambda_{2n})\Lambda_{2n} + (1 - \tan^2\Lambda_{2n}) \sin \Lambda_{2n} \cos \Lambda_{2n} + 2 \tan \Lambda_{2n} \sin^2\Lambda_{2n}].$$

The Λ_{1n} are eigenvalues, which are determined from implicit equation

$$\tan \Lambda_{1n} = \left(\frac{k_w \rho_w c_w}{k \rho c_p}\right)^{1/2} \cot \left(\Lambda_{1n} \frac{a}{d} \sqrt{\frac{\alpha_w}{\alpha}}\right).$$

When the A_{1n} are known, the A_{2n} are then obtained from

$$A_{2n} = \frac{a}{d}\left(\frac{\alpha_w}{\alpha}\right)^{1/2} A_{1n}$$

and the δ_n are given by

$$\delta_n = \frac{\sqrt{\alpha_w}}{d} A_{1n}.$$

Two numerical examples were carried out to illustrate the transient process. The fluid in the channel was taken to be air at 170°F with the following properties: $c_p = 0{\cdot}241$ Btu/lb °F, $\rho = 0{\cdot}0623$ lb/ft³, and $k = 0{\cdot}01735$ Btu/h ft °F. The channel walls were taken to be stainless steel (18 Cr–8 Ni) with the following properties: $c_w = 0{\cdot}11$ Btu/lb °F, $\rho_w = 488$ lb/ft³, and $k_w = 10$ Btu/h ft °F. The channel half-width was fixed at $a = \frac{1}{8}$ in and two wall thicknesses were used: $d = \frac{1}{32}$ and $\frac{1}{16}$ in. The heat conducted from the wall to the fluid is given as a function of time by the solid lines in Fig. 8. The series in equation (24) was evaluated for the first four terms, which brought the conduction curves back to sufficiently early times for the examples given here; and the curves were then extended approximately to the origin. These conduction curves give the exact transient solutions for the forced-convection problem during the initial transient period. This period ends when fluid that was originally outside of the heated section starts to affect the heat transfer at a given point, and the time at which this occurs is given by the characteristic curve passing through the origin of the X–Θ plane for $n = 0$. These times correspond to the minimum points of the curves for $d = 0$ obtained earlier and shown again in Fig. 8. For each axial position, the pure conduction curve in the complete solution terminates at the time corresponding to this minimum point.

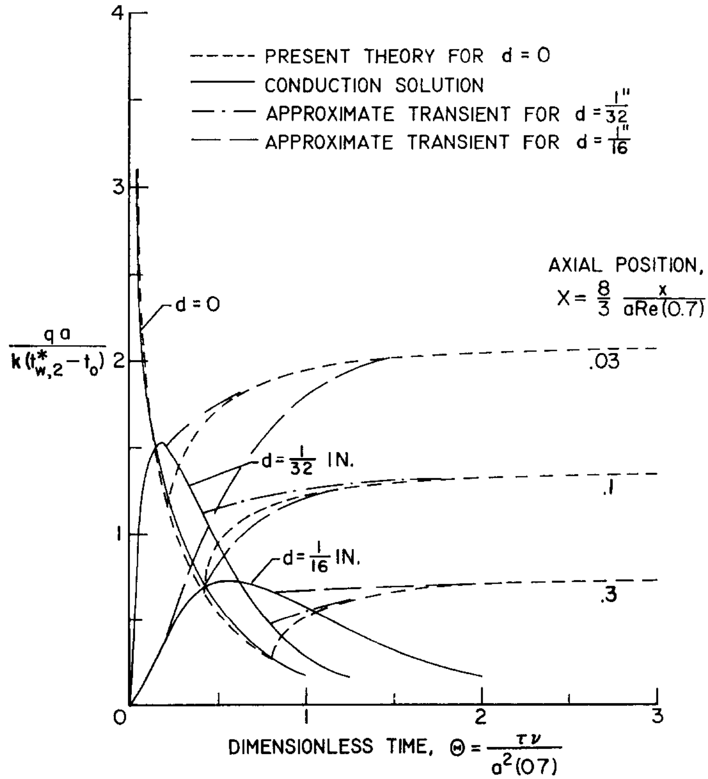

FIG. 8. Effect of channel wall thickness on transient heat flux from stainless steel walls to air for a step change in pressure gradient and outside wall temperature. $Pr = 0{\cdot}7$; initially: $u = u_1 = 0$, $t_w^* = t_{w,1}^* = t_0$; finally: $u = u_2$, $t_w^* = t_{w,2}^*$. Channel half width, $a = \frac{1}{8}$ in.

For large times, steady-state results must be achieved. For a finite wall resistance, the steady solution has been given in [6]. For the numerical examples given here, the wall resistance is sufficiently small that the final steady-state solution is uninfluenced by it, and the results for large times are the same as for the case with $d = 0$. For finite wall resistance we now have the solution for small and large times, and the time is known at which the initial period of pure conduction ends. For zero wall resistance we have the complete transient solution. Using this complete solution as a guide, the results for intermediate times are faired in for the finite wall resistance examples as shown in Fig. 8.

In some instances the presence of the wall has a considerable influence. Consider for example the curves for $X = 0{\cdot}1$. For $d = 0$ the heat flux starts from infinity, decreases with time to a minimum during the initial period of pure conduction, and then increases to the final steady value. When a wall of thickness $d = \frac{1}{32}$ in is introduced, the heat transfer to the fluid is initially zero. Then during the initial conduction process the heat flux rises rapidly to a maximum and then begins to decrease as the fluid temperature approaches the imposed outside wall temperature. For this example the fluid convection does not begin to have an influence until after the maximum in the pure conduction curve has been reached. Hence the heat transfer goes through a minimum and then the convection raises it to the final steady value. For $d = \frac{1}{16}$ in the heat comes through the wall so slowly that the heat flux by pure conduction is still increasing at the time that the convection begins to have an effect. Consequently, the heat flux rises from zero toward the steady value without passing through a minimum.

To have an indication of the actual time response in the channel, we note in Fig. 8 that, for the axial positions considered, the important transient effects occur during a dimensionless time Θ of about $1{\cdot}0$. Using $a = \frac{1}{8}$ in and $\nu = 22{\cdot}38 \times 10^{-5}$ ft^2/sec, this yields a value for τ of about $\frac{1}{3}$ sec. Hence for this example the transient period is quite short.

CONCLUDING REMARKS

Results have been presented for transient heat transfer arising from simultaneous changes with time of the channel wall temperature and fluid pumping pressure. In a previous paper [4] this type of transient solution was found for a one-dimensional velocity distribution in the channel, and the present work takes the two-dimensional velocity distribution into account. The present solutions exhibit the same general transient behavior as those in [4]. A method for investigating the effect of the heat-flow resistance of the channel walls has been demonstrated; and, as would be expected, this resistance can cause substantial changes in the heat transfer response. This is due to the fact that during the time delay required for the heat to flow through the wall the velocity is completing part of its transient.

This type of analysis is not restricted to abrupt pressure changes as considered here, but can also be carried out for other timewise pressure variations. Other types of boundary conditions can be treated for specified wall temperatures, such as one wall at uniform temperature and the other insulated. The analysis can be carried over in a straightforward fashion for the circular tube geometry.

ACKNOWLEDGEMENT

The authors would like to thank Miss Frances J. Schubert and Miss Eileen M. Norris who programed the machine computations for the examples presented here.

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